

Tail dependence convergence rate for the bivariate skew normal under the equal-skewness condition

Thomas Fung^{*1} and Eugene Seneta²

¹Department of Statistics, Macquarie University, NSW 2109, Australia

²School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

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Abstract

We derive the rate of decay of the tail dependence of the bivariate skew normal distribution under the equal-skewness condition $\theta_1 = \theta_2 = \theta$, say. The rate of convergence depends on whether $\theta > 0$ or $\theta < 0$. The latter case gives rate asymptotically identical with the case $\theta = 0$. The asymptotic behaviour of the quantile function for the univariate skew normal is part of the theoretical development.

Keywords: Asymptotic tail dependence coefficient; bivariate skew normal distribution; convergence rate; quantile function.

1 Introduction

The coefficient of lower tail dependence of a random vector $\mathbf{X} = (X_1, X_2)^T$ with marginal inverse distribution function F_1^{-1} and F_2^{-1} is defined as

$$\lambda_L = \lim_{u \rightarrow 0^+} \lambda_L(u), \quad \text{where} \quad \lambda_L(u) = P(X_1 \leq F_1^{-1}(u) | X_2 \leq F_2^{-1}(u)). \quad (1)$$

\mathbf{X} is said to have asymptotic lower tail dependence if λ_L exists and is positive. If $\lambda_L = 0$, then \mathbf{X} is said to be asymptotically independent in the lower tail. This quantity provides insight on the tendency for the distribution to generate joint extreme event since it measures the strength of dependence (or association) in the lower tails of a bivariate

^{*}Corresponding Author. Honorary Associate, University of Sydney. Email address: thomas.fung@mq.edu.au.

distribution. If the marginal distributions of these random variables are continuous, then from (1), it follows that $\lambda_L(u)$ can be expressed in terms of the copula of \mathbf{X} , $C(u_1, u_2)$, as

$$\lambda_L(u) = \frac{P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u))}{P(X_2 \leq F_2^{-1}(u))} = \frac{C(u, u)}{u}. \quad (2)$$

Ramos and Ledford (2009), continuing the work of Ledford and Tawn (1997), studied intensively a family of bivariate distributions (which they characterised) which satisfied in particular the condition

$$\lambda_L(u) = u^{\frac{1}{\alpha}-1} L(u) \quad (3)$$

where $L(u)$ is a slowly varying function (SVF) as $u \rightarrow 0^+$, and $\alpha \in (0, 1]$, so that, in fact, the value of α could be used for comparison of the degree of tail dependence structure between members of the family. The standard bivariate extreme value models correspond to $\alpha = 1$.

Hua and Joe (2011) developed this idea further and defined $\kappa = 1/\alpha$ in (3) as the (lower) tail order of a copula. The case $1 < \kappa < 2$ is termed as intermediate tail dependence as it represents the copula has some level of positive dependence in the tail but not as strong as tail dependence with $\lambda_L = 0$. The tail order κ can be used to assess tail dependence strength when $\lambda_L = 0$.

The standard bivariate normal with correlation coefficient $-1 < \rho < 1$ corresponds to $\alpha = \frac{1+\rho}{2}$ in (3), and hence is an instance of intermediate tail dependence. A recent manifestation of this more or less known result is in Fung and Seneta (2011), where it is shown that

$$L(u) \sim 2\sqrt{\frac{1+\rho}{1-\rho}}(-4\pi \log u)^{-\frac{\rho}{1+\rho}}, \quad \text{as } u \rightarrow 0^+.$$

The bivariate skew normal distribution was introduced in Azzalini and Dalla Valle (1996) (which is discussed further in Azzalini and Capitanio (1999)). A random vector \mathbf{X} is said to have a bivariate skew normal distribution, denoted as $\mathbf{X} \sim SN_2(\boldsymbol{\theta}, R)$, if the probability density of \mathbf{X} is

$$f(\mathbf{x}) = 2\phi_2(\mathbf{x}, R)\Phi(\boldsymbol{\theta}^T \mathbf{x}), \quad (4)$$

where $\phi(\cdot, R)$ is density of a bivariate normal distribution with mean $\mathbf{0}$ and correlation matrix R and $\Phi(\cdot)$ is the cdf of a univariate standard normal distribution. The correlation matrix R and skew vector $\boldsymbol{\theta}$ are defined as $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, with $-1 < \rho < 1$ and $\boldsymbol{\theta} = (\theta_1, \theta_2)^T \in \mathbb{R}^2$ respectively. Obviously, the (symmetric) normal is obtained as special case when $\boldsymbol{\theta} = \mathbf{0}$.

The results of Lysenko, Roy and Waeber (2009), Bortot (2010) and Padoan (2011) show that the skew normal distribution is tail independent, that is: $\lambda_L = \lim_{u \rightarrow 0^+} \lambda_L(u) = 0$.

The focus of this current note is thus to consider (3) in the setting of the skew normal distribution.

We were motivated to do this not only by interest in generalizing to bivariate skew normal the result for the bivariate normal, but also by the following. Fung and Seneta (2014) chose to view results such as (3) as a rate of convergence result to the limit value $\lambda_L = 0$. The skew t distribution is tail dependent i.e. $\lambda_L > 0$ and Fung and Seneta (2014) obtained for it the rate of convergence result:

$$\lambda_L(u) - \lambda_L = \mathcal{K}(\eta, R, \boldsymbol{\theta}) u^{\frac{2}{\eta}} + O(u^{\frac{4}{\eta}}) \quad (5)$$

as $u \rightarrow 0^+$, where $\mathcal{K}(\eta, R, \boldsymbol{\theta})$ is a constant which depends on the distribution's degrees of freedom η , scale matrix R and skewness vector $\boldsymbol{\theta}$. This generalised the results of Manner and Segers (2011) and Chicheportiche and Bouchaud (2012), who had taken the rate of convergence standpoint and showed its practical importance, for the (symmetric) Student's t distribution.

Now, this skew t approaches the skew normal and there are cases where λ_L and $\mathcal{K}(\eta, R, \boldsymbol{\theta}) \rightarrow 0$ as $\eta \rightarrow \infty$. But it is clear that the expression for $\lambda_L(u)$ deriving from (5), in the limit does not seem to provide useful information about an expression for convergence rate for the limit. Our treatment of convergence rate for the skew t case had worked in a unified way when θ_1 was not necessarily the same as θ_2 . For the skew-normal we confined ourselves for this paper to equi-skewness: that is $\theta_1 = \theta_2 = \theta$. Even so, we found that our approach provided qualitatively and quantitatively different results in the cases $\theta > 0$ and $\theta < 0$, the latter case being asymptotically (apart from a constant multiplier) identical to the symmetric bivariate normal case $\theta = 0$. The two cases required quite different approaches.

Upper tail dependence behaviour is expressed immediately from our result on lower tail dependence behaviour, using the device in Fung and Seneta (2014), Section 5.

The remainder of this paper is set out as follows. In Section 2, we derive the asymptotic behaviour of the quantile function for the skew normal which is needed for our subsequent proof. In Section 3, we derive the rate of convergence in the form of (3) for the skew normal distribution.

2 Asymptotic behaviour of the quantile function

Under the equal skewness condition i.e. $\theta_1 = \theta_2 = \theta$, both X_1 and X_2 will have the same marginal distribution. Without loss of generality, we shall focus on X_1 . The marginal density for X_1 is

$$f_{X_1}(x_1) = 2\phi(x_1)\Phi(\lambda x_1), \quad \text{for } x_1 \in \mathbb{R}, \quad (6)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the univariate standard normal and

$$\lambda = \frac{\theta(1 + \rho)}{\sqrt{1 + \theta^2(1 - \rho^2)}}. \quad (7)$$

This means that both X_1 and X_2 have a univariate skew normal distribution with skewness parameter λ i.e. $X_i \sim SN(\lambda)$, $i = 1, 2$.

Using Lemma 2 of Capitanio (2010), we have the inequalities for $P(X_1 \leq z)$, for $z < 0$,

$$\begin{aligned} & \frac{1}{\pi} e^{-\frac{1}{2}(1+\lambda^2)z^2} \left[\frac{1}{\lambda(1+\lambda^2)} z^{-2} - \left(\frac{2}{\lambda(1+\lambda^2)} + \frac{1}{\lambda^3(1+\lambda^2)} \right) z^{-4} \right] \\ & < P(X_1 \leq z) < \frac{1}{\pi} e^{-\frac{1}{2}(1+\lambda^2)z^2} \left[\frac{1}{\lambda(1+\lambda^2)} z^{-2} \right] \end{aligned} \quad (8)$$

when $\lambda > 0$ and

$$\begin{aligned} & \frac{2}{\sqrt{2\pi}} e^{-z^2/2} \left[|z|^{-1} - \sqrt{\frac{2}{\pi}} \frac{1}{|\lambda|(1+\lambda^2)} z^{-2} e^{-z^2\lambda^2/2} - |z|^{-3} \right] \\ & < P(X_1 \leq z) < \frac{2}{\sqrt{2\pi}} e^{-z^2/2} \left[|z|^{-1} - \sqrt{\frac{2}{\pi}} \frac{1}{|\lambda|(1+\lambda^2)} z^{-2} e^{-z^2\lambda^2/2} \right. \\ & \quad \left. + \sqrt{\frac{2}{\pi}} |z|^{-4} e^{-z^2\lambda^2/2} \left(\frac{2}{|\lambda|(1+\lambda^2)^2} + \frac{1}{|\lambda|^3(1+\lambda^2)} \right) \right] \end{aligned} \quad (9)$$

when $\lambda < 0$. This means that

$$F_1(z) = P(X_1 \leq z) \sim \begin{cases} \frac{1}{\pi\lambda(1+\lambda^2)} |z|^{-2} e^{-\frac{1}{2}(1+\lambda^2)z^2}, & \text{for } \lambda > 0; \\ \sqrt{\frac{2}{\pi}} |z|^{-1} e^{-z^2/2}, & \text{for } \lambda < 0 \end{cases} \quad \text{as } z \rightarrow -\infty. \quad (10)$$

Recall, for comparison, the cdf of standard normal has the following asymptotic behaviour (see for instance Feller (1968) Chapter VII Lemma 2):

$$\Phi(z) \sim \frac{1}{\sqrt{2\pi}} |z|^{-1} e^{-\frac{1}{2}z^2}, \quad \text{as } z \rightarrow -\infty \quad (11)$$

and neither of the expressions on the right-hand side of (10) reduces to (11) by letting $\lambda \rightarrow 0$.

The quantile results are summarised into the following theorem. The asymptotic expressions on the right are given in form convenient for the sequel.

Theorem 1. *Let $X_1 \sim SN(\lambda)$, then*

$$F_1^{-1}(u) \sim y(u) = \begin{cases} -\sqrt{-\frac{2}{1+\lambda^2} \log(-2\pi\lambda u \log(2\pi\lambda u))}, & \text{if } \lambda > 0; \\ -\sqrt{-2 \log\left(\frac{u}{2} \sqrt{-4\pi \log\left(\frac{u}{2} \sqrt{2\pi}\right)}\right)}, & \text{if } \lambda < 0, \end{cases} \quad \text{as } u \rightarrow 0^+.$$

Proof. In order to find the asymptotic behaviour of the quantile functions $F_1^{-1}(u)$ as $u \rightarrow 0^+$, we shall use Theorem 1 of Fung and Seneta (2011), which requires to find $-y(u)$, a slowly varying function (SVF) as $u \rightarrow 0^+$, such that $F_1(y(u))/u \rightarrow 1$ as $u \rightarrow 0^+$. For

then, according to that theorem, $F_1^{-1}(u) \sim y(u)$ as $u \rightarrow 0^+$. To proceed, we first note that both asymptotic expression on the right of (10) are of the form

$$u = a|g(u)|^b e^{-c|g(u)|^d}, \quad (12)$$

where $a, c, d > 0$, $b < 0$. To solve for $g(u)$ in (12) as $u \rightarrow 0^+$ we use the Lambert W function. The function itself is defined as the solution w to $z = we^w$ for $z > 0$: that is, as the inverse function of the positive continuous and increasing function xe^x , $x > 0$. Moreover, the asymptotic behaviour of Lambert W function is given by

$$W(z) = \log z - \log \log z + O\left(\frac{\log \log z}{\log z}\right), \quad (13)$$

as $z \rightarrow \infty$ so $W(z) \rightarrow \infty$ as $z \rightarrow \infty$; see Corless *et al.* (1996). From (12),

$$u = a|g(u)|^b e^{-c|g(u)|^d} \Rightarrow \frac{cd}{|b|}|g(u)|^d e^{\frac{cd}{|b|}|g(u)|^d} = \frac{cd}{|b|} \left(\frac{a}{u}\right)^{d/|b|},$$

so the LHS of the last expression is in the form of we^w . Thus

$$\frac{cd}{|b|}|g(u)|^d = W\left(\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}\right) \quad (14)$$

where $W(\cdot)$ is the Lambert W function. As $u \rightarrow 0^+$, $u^{-\frac{d}{|b|}} \rightarrow \infty$ and we can combine (13) with (14) to get

$$\begin{aligned} \frac{cd}{|b|}|g(u)|^d &= \log\left(\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}\right) - \log \log\left(\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}\right) + O\left(\frac{\log \log\left(\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}\right)}{\log\left(\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}\right)}\right) \\ \Rightarrow |g(u)| &= \left\{ \frac{|b|}{cd} \left[\log\left(\frac{\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}}{\log\left(\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}\right)}\right) + O\left(\frac{\log \log\left(\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}\right)}{\log\left(\frac{cd}{|b|} \left(\frac{a}{u}\right)^{\frac{d}{|b|}}\right)}\right) \right] \right\}^{\frac{1}{d}}, \quad (15) \end{aligned}$$

as $u \rightarrow 0^+$.

Comparing the expression on the right of (10) with (12), we have $a = \frac{1}{\pi\lambda(1+\lambda^2)}$, $b = -2$, $c = \frac{1}{2}(1+\lambda^2)$ and $d = 2$ so that $\frac{cd}{|b|} = \frac{1+\lambda^2}{2}$ for $\lambda > 0$; and $a = \sqrt{2/\pi}$, $b = -1$, $c = 1/2$ and $d = 2$ so that $\frac{cd}{|b|} = 1$ for $\lambda < 0$. Substitute these constants into (15) to get

$$\begin{aligned} g(u) &= -\left\{ \frac{2}{(1+\lambda^2)} \left[\log\left(\frac{\frac{(1+\lambda^2)}{2} \left(\frac{1}{\pi\lambda(1+\lambda^2)u}\right)}{\log\left(\frac{(1+\lambda^2)}{2} \left(\frac{1}{\pi\lambda(1+\lambda^2)u}\right)}\right)}\right) \right. \right. \\ &\quad \left. \left. + O\left(\frac{\log \log\left(\frac{(1+\lambda^2)}{2} \left(\frac{1}{\pi\lambda(1+\lambda^2)u}\right)}\right)}{\log\left(\frac{(1+\lambda^2)}{2} \left(\frac{1}{\pi\lambda(1+\lambda^2)u}\right)}\right)}\right) \right] \right\}^{\frac{1}{2}} \\ &\sim -\sqrt{-\frac{2}{1+\lambda^2} \log(-2\pi\lambda u \log(2\pi\lambda u))} \quad (16) \end{aligned}$$

$$\sim -\sqrt{-\frac{2}{1+\lambda^2}\log u}, \quad (17)$$

as $u \rightarrow 0^+$ for $\lambda > 0$; and

$$\begin{aligned} g(u) &= -\left\{ \log \left(\frac{\left(\frac{1}{u}\sqrt{\frac{2}{\pi}}\right)^2}{\log \left(\frac{1}{u}\sqrt{\frac{2}{\pi}}\right)^2} \right) + O \left(\frac{\log \log \left(\frac{1}{u}\sqrt{\frac{2}{\pi}}\right)^2}{\log \left(\frac{1}{u}\sqrt{\frac{2}{\pi}}\right)^2} \right) \right\}^{\frac{1}{2}} \\ &\sim -\sqrt{-2 \log \left(\frac{u}{2} \sqrt{-4\pi \log \left(\frac{u}{2} \sqrt{2\pi} \right)} \right)} \end{aligned} \quad (18)$$

$$\sim -\sqrt{-2 \log u}, \quad (19)$$

as $u \rightarrow 0^+$ for $\lambda < 0$. Now set $y(u)$ as the right-hand side of (16) and (18) in the respective cases $\lambda > 0$ and $\lambda < 0$. It is clear that $-y(u)$ is SVF as $u \rightarrow 0^+$ from (17) and (19), and since $y(u) \rightarrow -\infty$ as $u \rightarrow 0^+$, that $F_1(y(u))/u \rightarrow 1$, using (16), (18) and the right-hand side of (10). \square

Notice (and compare with (18)) that the asymptotic behaviour of the quantile function for the standard normal is

$$\begin{aligned} \Phi^{-1}(u) &\sim -\sqrt{-2 \log(u \sqrt{-4\pi \log u})} \\ &\sim -\sqrt{-2 \log u}, \quad \text{as } u \rightarrow 0^+. \end{aligned} \quad (20)$$

(See Fung and Seneta (2011) where the above methodology is used to obtain (20).)

3 Main result

Similarly to the univariate cdf (and the corresponding quantile function), the rate of convergence to zero of the lower and upper tail dependence function depends heavily on whether $\theta > 0$ or $\theta < 0$. The results are summarised into the following theorem.

Theorem 2. *Let $\mathbf{X} \sim SN_2(\boldsymbol{\theta}, R)$ with $\theta_1 = \theta_2 = \theta$. As $u \rightarrow 0^+$,*

(a) *if $\theta > 0$,*

$$\lambda_L(u) \sim u^{\beta^2} \frac{\alpha^3}{\pi \lambda^4 \beta (1 + \beta^2)^2} \sqrt{\frac{2}{\pi}} (2\pi \lambda)^{1+\beta^2} \left(\frac{1+\lambda^2}{2}\right)^{\frac{3}{2}} [-\log u]^{\beta^2 - \frac{1}{2}} \quad (21)$$

$$\text{with } \lambda = \frac{\theta(1+\rho)}{\sqrt{1+\theta^2(1-\rho^2)}}, \alpha = \frac{\theta(1+\rho)}{\sqrt{1+2\theta^2(1+\rho)}} \text{ and } \beta = \sqrt{\frac{(1-\rho)(1+2\theta^2(1+\rho))}{1+\rho}};$$

(b) *if $\theta < 0$,*

$$\lambda_L(u) \sim u^{\frac{1-\rho}{1+\rho}} \times \frac{1+\rho}{2} \sqrt{\frac{1+\rho}{1-\rho}} (-\pi \log u)^{-\frac{\rho}{1+\rho}}. \quad (22)$$

Proof. The proof will be divided into two parts depends on whether $\theta > 0$ or $\theta < 0$. We will first consider the case $\theta > 0$ and mixture representation forms the basis of this proof.

For a given pair of $(\boldsymbol{\theta}, \Sigma)$ in (4), it has been shown in Azzalini and Capitanio (1999) that there is a pair of $(\boldsymbol{\alpha}, \Psi)$ such that \mathbf{X} can be represented as a normal mean mixture by

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\alpha}V + \mathbf{Z}, \quad (23)$$

where

$$\boldsymbol{\alpha} = \frac{R\boldsymbol{\theta}}{\sqrt{1 + \boldsymbol{\theta}^T R \boldsymbol{\theta}}} = \left(\frac{\theta(1+\rho)}{\sqrt{1+2\theta^2(1+\rho)}}, \frac{\theta(1+\rho)}{\sqrt{1+2\theta^2(1+\rho)}} \right)^T = (\alpha, \alpha)^T; \quad (24)$$

and

$$\Psi = R - (1 + \boldsymbol{\theta}^T R \boldsymbol{\theta})^{-1} R \boldsymbol{\theta} \boldsymbol{\theta}^T R = \begin{pmatrix} \frac{1+\theta^2(1-\rho^2)}{1+2\theta^2(1+\rho)} & \frac{\rho-\theta^2(1-\rho^2)}{1+2\theta^2(1+\rho)} \\ \frac{\rho-\theta^2(1-\rho^2)}{1+2\theta^2(1+\rho)} & \frac{1+\theta^2(1-\rho^2)}{1+2\theta^2(1+\rho)} \end{pmatrix} = \begin{pmatrix} \frac{\alpha^2}{\lambda^2} & \frac{\rho-\theta^2(1-\rho^2)}{1+2\theta^2(1+\rho)} \\ \frac{\rho-\theta^2(1-\rho^2)}{1+2\theta^2(1+\rho)} & \frac{\alpha^2}{\lambda^2} \end{pmatrix}.$$

Note that $\theta > 0$ implies that $\alpha > 0$. This parametrisation also satisfies the condition that Ψ is symmetric and positive definite. $\mathbf{Z} \sim N(\mathbf{0}, \Psi)$ is the bivariate normal with $\mathbf{0}$ mean and covariance matrix Ψ , and $V \sim$ Half Normal with pdf

$$f_V(v) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-\frac{v^2}{2}}, & \text{if } v > 0; \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

That is $V = |W|$, with $W \sim N(0, 1)$. V is assumed to be distributed independently of \mathbf{Z} . Obviously, when $\boldsymbol{\theta} = \mathbf{0}$ (and equivalently $\boldsymbol{\alpha} = \mathbf{0}$), we have the usual (symmetric) multivariate normal as special case for the distribution of \mathbf{X} .

Continuing, when $\theta > 0$

$$\begin{aligned} & P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u)) \\ &= P(X_1 \leq x, X_2 \leq x), \quad \text{where } x = x(u) = F_1^{-1}(u) = F_2^{-1}(u) \\ &= E_V(P(\alpha V + Z_1 \leq x, \alpha V + Z_2 \leq x)) \\ &= E_V \left(P \left(\frac{Z_1}{\alpha/\lambda} \leq \frac{x - \alpha V}{\alpha/\lambda}, \frac{Z_2}{\alpha/\lambda} \leq \frac{x - \alpha V}{\alpha/\lambda} \right) \right) \\ &= E_V \left(P \left(Z_1^* \leq \frac{x - \alpha V}{\alpha/\lambda}, Z_2^* \leq \frac{x - \alpha V}{\alpha/\lambda} \right) \right) \\ &= E_V \left(P \left(\max(Z_1^*, Z_2^*) \leq \frac{x - \alpha V}{\alpha/\lambda} \right) \right) \end{aligned} \quad (26)$$

where we define

$$\mathbf{Z}^* = (Z_1^*, Z_2^*)^T = \left(\frac{Z_1}{\alpha/\lambda}, \frac{Z_2}{\alpha/\lambda} \right)^T \sim N(\mathbf{0}, \begin{pmatrix} 1 & \frac{\rho-\theta^2(1-\rho^2)}{1+\theta^2(1-\rho^2)} \\ \frac{\rho-\theta^2(1-\rho^2)}{1+\theta^2(1-\rho^2)} & 1 \end{pmatrix}).$$

Using the results from Roberts (1966) and Loperfido (2002), we know that $Z_{(2)}^* = \max(Z_1^*, Z_2^*) \sim SN(\beta)$ i.e. a univariate skew normal distribution with skewness

$$\beta = \sqrt{\frac{1 - \frac{\rho - \theta^2(1-\rho^2)}{1+\theta^2(1-\rho^2)}}{1 + \frac{\rho - \theta^2(1-\rho)}{1+\theta^2(1-\rho^2)}}} = \sqrt{\frac{(1-\rho)(1+2\theta^2(1+\rho))}{(1+\rho)}}. \quad (27)$$

If we combine this fact with the Capitanio bounds in (8), (26) becomes

$$\begin{aligned} & \int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}(1+\beta^2)\left(\frac{x-\alpha v}{\alpha/\lambda}\right)^2} \times \frac{1}{\beta(1+\beta^2)} \left(\frac{x-\alpha v}{\alpha/\lambda}\right)^{-2} f_V(v) dv \\ & - \int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}(1+\beta^2)\left(\frac{x-\alpha v}{\alpha/\lambda}\right)^2} \left(\frac{2}{\beta(1+\beta^2)} + \frac{1}{\beta^3(1+\beta^2)}\right) \left(\frac{x-\alpha v}{\alpha/\lambda}\right)^{-4} f_V(v) dv \\ & < E_V \left(P \left(Z_{(2)}^* \leq \frac{x-\alpha V}{\alpha/\lambda} \right) \right) \\ & < \int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}(1+\beta^2)\left(\frac{x-\alpha v}{\alpha/\lambda}\right)^2} \frac{1}{\beta(1+\beta^2)} \left(\frac{x-\alpha v}{\alpha/\lambda}\right)^{-2} f_V(v) dv, \end{aligned} \quad (28)$$

which suggests that we need to compare the upper bound in (28) with

$$|x|^{-3} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)x^2},$$

and we will consider

$$\begin{aligned} & \frac{\int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}(1+\beta^2)\left(\frac{x-\alpha v}{\alpha/\lambda}\right)^2} \times \frac{1}{\beta(1+\beta^2)} \left(\frac{x-\alpha v}{\alpha/\lambda}\right)^{-2} f_V(v) dv}{|x|^{-3} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)x^2}} \\ & = |x| \int_0^\infty \frac{1}{\pi} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)(-2\alpha x v + \alpha^2 v^2)} \times \frac{\alpha^2}{\lambda^2 \beta(1+\beta^2)} \frac{1}{(1 + \alpha \frac{v}{|x|})^2} f_V(v) dv \end{aligned}$$

Applying integration by parts:

$$\begin{aligned} & = \frac{\alpha^2 |x|}{\pi \lambda^2 \beta(1+\beta^2)} \left\{ \left[\frac{\alpha}{\lambda^2(1+\beta^2)x} e^{\frac{\lambda^2}{\alpha}(1+\beta^2)xv} \times \frac{e^{-\lambda^2(1+\beta^2)(\frac{v^2}{2})} f_V(v)}{(1 + \alpha \frac{v}{|x|})^2} \right]_0^\infty \right. \\ & \quad \left. - \int_0^\infty \frac{\alpha}{\lambda^2(1+\beta^2)x} e^{\frac{\lambda^2}{\alpha}(1+\beta^2)xv} \times \frac{d}{dv} \left(\frac{e^{-\lambda^2(1+\beta^2)(\frac{v^2}{2})} f_V(v)}{(1 + \alpha \frac{v}{|x|})^2} \right) dv \right\} \\ & = \frac{\alpha^2}{\pi \lambda^2 \beta(1+\beta^2)} \left\{ \frac{\alpha f_V(0)}{\lambda^2(1+\beta^2)} + \int_0^\infty \frac{\alpha}{\lambda^2(1+\beta^2)} e^{\frac{\lambda^2}{\alpha}(1+\beta^2)xv} \right. \\ & \quad \left. \times \frac{d}{dv} \left(\frac{e^{-\lambda^2(1+\beta^2)(\frac{v^2}{2})} f_V(v)}{(1 + \alpha \frac{v}{|x|})^2} \right) dv \right\} \end{aligned} \quad (29)$$

As

$$\begin{aligned}
& \left| \frac{d}{dv} \left(\frac{e^{-\lambda^2(1+\beta^2)(\frac{v^2}{2})} f_V(v)}{(1 + \alpha \frac{v}{|x|})^2} \right) \right| \\
&= \left| \frac{-2}{(1 + \alpha \frac{v}{|x|})^3} \frac{\alpha}{|x|} e^{-\lambda^2(1+\beta^2)(\frac{v^2}{2})} f_V(v) + \frac{1}{(1 + \alpha \frac{v}{|x|})^2} [-\lambda^2(1 + \beta^2)v] \right. \\
&\quad \left. \times e^{-\lambda^2(1+\beta^2)(\frac{v^2}{2})} f_V(v) + \frac{e^{-\lambda^2(1+\beta^2)(\frac{v^2}{2})}}{(1 + \alpha \frac{v}{|x|})^2} \frac{d}{dv} f_V(v) \right| \\
&\leq e^{-\lambda^2(1+\beta^2)(\frac{v^2}{2})} \left(2\alpha f_V(v) + \lambda^2(1 + \beta^2)v f_V(v) + \left| \frac{d}{dv} f_V(v) \right| \right) < \infty,
\end{aligned}$$

and $\frac{d}{dv} f_V(v) = \sqrt{\frac{2}{\pi}}(-v)e^{-v^2/2}$, we only need

$$\int_0^\infty v e^{-\xi v^2} dv < \infty$$

for some $\xi > 0$ to have dominated convergence in (29) and the condition is obviously true. Thus,

$$\lim_{x \rightarrow -\infty} \frac{\int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}(1+\beta^2)(\frac{x-\alpha v}{\alpha/\lambda})^2} \times \frac{1}{\beta(1+\beta^2)} \left(\frac{x-\alpha v}{\alpha/\lambda} \right)^{-2} f_V(v) dv}{|x|^{-3} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)x^2}} = \frac{\alpha^3 f_V(0)}{\pi \lambda^4 \beta (1 + \beta^2)^2},$$

which also implies that

$$\begin{aligned}
& \int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}(1+\beta^2)(\frac{x-\alpha v}{\alpha/\lambda})^2} \times \frac{1}{\beta(1+\beta^2)} \left(\frac{x-\alpha v}{\alpha/\lambda} \right)^{-2} f_V(v) dv \\
& \sim \frac{\alpha^3}{\pi \lambda^4 \beta (1 + \beta^2)^2} \sqrt{\frac{2}{\pi}} |x|^{-3} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)x^2}.
\end{aligned} \tag{30}$$

As the first term in the lower bound is the same as the upper bound in (28), we will now consider the higher order term in the lower bound in the form

$$\begin{aligned}
& \frac{\int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}(1+\beta^2)(\frac{x-\alpha v}{\alpha/\lambda})^2} \times \left(\frac{2}{\beta(1+\beta^2)} + \frac{1}{\beta^3(1+\beta^2)} \right) \left(\frac{x-\alpha v}{\alpha/\lambda} \right)^{-4} f_V(v) dv}{|x|^{-5} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)x^2}} \\
&= |x| \int_0^\infty \frac{1}{\pi} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)(-2\alpha v x + \alpha^2 v^2)} \times \left(\frac{2}{\beta(1+\beta^2)} + \frac{1}{\beta^3(1+\beta^2)} \right) \left(\frac{\alpha^4}{\lambda^4} \right) \frac{f_V(v)}{(1 + \alpha \frac{v}{|x|})^4} dv \\
&\rightarrow \frac{\alpha^5}{\pi \lambda^6 (1 + \beta^2)} \left(\frac{2}{\beta(1+\beta^2)} + \frac{1}{\beta^3(1+\beta^2)} \right) f_V(0)
\end{aligned}$$

after integration by parts and dominated convergence again as $x \rightarrow -\infty$. Thus

$$\int_0^\infty \frac{1}{\pi} e^{-\frac{1}{2}(1+\beta^2)(\frac{x-\alpha v}{\alpha/\lambda})^2} \times \left(\frac{2}{\beta(1+\beta^2)} + \frac{1}{\beta^3(1+\beta^2)} \right) \left(\frac{x-\alpha v}{\alpha/\lambda} \right)^{-4} f_V(v) dv$$

$$= O\left(|x|^{-5} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)x^2}\right) \quad (31)$$

as $x \rightarrow -\infty$. By combining (28), (30) and (31), we have

$$P(X_1 \leq x, X_2 \leq x) \sim \frac{\alpha^3}{\pi\lambda^4\beta(1+\beta^2)^2} \sqrt{\frac{2}{\pi}} |x|^{-3} e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2)x^2},$$

as $x \rightarrow -\infty$ and by substituting in (16) we have

$$\begin{aligned} & P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u)) \\ & \sim \frac{\alpha^3}{\pi\lambda^4\beta(1+\beta^2)^2} \sqrt{\frac{2}{\pi}} \left(-\frac{2}{1+\lambda^2} \log(-2\pi\lambda u \log(2\pi\lambda u)) \right)^{-\frac{3}{2}} \\ & \quad \times e^{-\frac{\lambda^2}{2\alpha^2}(1+\beta^2) \left(-\frac{2}{1+\lambda^2} \log(-2\pi\lambda u \log(2\pi\lambda u)) \right)} \\ & \sim \frac{\alpha^3}{\pi\lambda^4\beta(1+\beta^2)^2} \sqrt{\frac{2}{\pi}} (2\pi\lambda)^{\frac{\lambda^2(1+\beta^2)}{\alpha^2(1+\lambda^2)}} \left(\frac{1+\lambda^2}{2} \right)^{\frac{3}{2}} [-\log u]^{\frac{\lambda^2(1+\beta^2)}{\alpha^2(1+\lambda^2)} - \frac{3}{2}} u^{\frac{\lambda^2(1+\beta^2)}{\alpha^2(1+\lambda^2)}} \\ & = \frac{\alpha^3}{\pi\lambda^4\beta(1+\beta^2)^2} \sqrt{\frac{2}{\pi}} (2\pi\lambda)^{1+\beta^2} \left(\frac{1+\lambda^2}{2} \right)^{\frac{3}{2}} [-\log u]^{\beta^2 - \frac{1}{2}} \times u^{1+\beta^2}. \end{aligned}$$

as

$$\begin{aligned} \alpha^2(1+\lambda^2) &= \frac{\theta^2(1+\rho)^2}{1+2\theta^2(1+\rho)} \left(1 + \frac{\theta^2(1+\rho^2)}{1+\theta^2(1-\rho^2)} \right) \\ &= \frac{\theta^2(1+\rho)^2}{1+2\theta^2(1+\rho)} \left(\frac{1+2\theta^2(1+\rho)}{1+\theta^2(1-\rho^2)} \right) = \lambda^2 \end{aligned}$$

since $\lambda = \frac{\theta(1+\rho)}{\sqrt{1+\theta^2(1-\rho^2)}}$ from (7) and $\alpha = \frac{\theta(1+\rho)}{\sqrt{1+2\theta^2(1+\rho)}}$ from (24). Finally, from (1)

$$\begin{aligned} & P(X_1 \leq F_1^{-1}(u) | X_2 \leq F_2^{-1}(u)) = \frac{P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u))}{u} \\ & \sim \frac{\alpha^3}{\pi\lambda^4\beta(1+\beta^2)^2} \sqrt{\frac{2}{\pi}} (2\pi\lambda)^{1+\beta^2} \left(\frac{1+\lambda^2}{2} \right)^{\frac{3}{2}} [-\log u]^{\beta^2 - \frac{1}{2}} \times u^{\beta^2}, \end{aligned}$$

which is (21) and part (a) of the proof is now completed.

Next, we consider the case $\theta < 0$. In this part of the proof, we proceed by noting that from (2) that $C(u, u) = \int_0^u \frac{dC(z, z)}{dz} dz$, so that if $\frac{dC(z, z)}{dz} = z^\tau L(z)$, $\tau > 0$ where $L(z)$ is a slowly varying function as $z \rightarrow 0^+$, then by (applying with suitable transformation to regular variation at 0) a result of de Haan (see Seneta (1976), p. 87), we obtain

$$\frac{C(u, u)}{u} = \frac{1}{u} \int_0^u \frac{dC(z, z)}{dz} dz \sim \frac{u^\tau L(u)}{\tau + 1}, \quad u \rightarrow 0^+. \quad (32)$$

Therefore, it is sufficient for us to find a value of $\tau > 0$ which satisfies $\frac{dC(u, u)}{du} = u^\tau L(u)$, for some slowly varying function $L(u)$, as $u \rightarrow 0^+$, so that (32) holds.

From (1), using L'Hôpital's rule and some well established basic properties of the derivative of copulas (see Nelsen (2006), pp.13, 41), we have

$$\begin{aligned}\frac{dC(u, u)}{du} &= P(X_2 \leq F_2^{-1}(u)|X_1 = F_1^{-1}(u)) + P(X_1 \leq F_1^{-1}(u)|X_2 = F_2^{-1}(u)) \\ &= 2P(X_2 \leq x|X_1 = x),\end{aligned}\tag{33}$$

by letting $x = F_1^{-1}(u) = F_2^{-1}(u)$. By using (4) and (6), the last expression can be written as

$$\begin{aligned}2P(X_2 \leq x|X_1 = x) &= 2 \int_{-\infty}^x \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(x_2-\rho x)^2}{2(1-\rho^2)}} \frac{\Phi(\theta x + \theta x_2)}{\Phi(\lambda x)} dx_2 \\ &= 2 \int_{-\infty}^{\sqrt{\frac{1-\rho}{1+\rho}}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \frac{\Phi(\theta\sqrt{1-\rho^2}z + \theta(1+\rho)x)}{\Phi(\lambda x)} dz, \quad \text{by letting } z = \frac{x_2 - \rho x}{\sqrt{1-\rho^2}}.\end{aligned}$$

Since $\theta < 0$, we have $\lambda = \frac{\theta(1+\rho)}{\sqrt{1+\theta^2(1-\rho^2)}} < 0$, which implies that $\Phi(\lambda x) \rightarrow 1$ as $x \rightarrow -\infty$, that is, when $u \rightarrow 0^+$. Moreover,

$$\begin{aligned}-\infty < z < \sqrt{\frac{1-\rho}{1+\rho}}x &\Rightarrow \theta(1-\rho)x < \theta\sqrt{1-\rho^2}z < \infty, \quad \text{as } \theta < 0; \\ &\Rightarrow 2\theta x < \theta\sqrt{1-\rho^2}z + \theta(1+\rho)x < \infty \\ &\Rightarrow \Phi(2\theta x) < \Phi(\theta\sqrt{1-\rho^2}z + \theta(1+\rho)x) < 1.\end{aligned}$$

As a result,

$$2 \int_{-\infty}^{\sqrt{\frac{1-\rho}{1+\rho}}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \frac{\Phi(2\theta x)}{\Phi(\lambda x)} dz < 2P(X_2 \leq x|X_1 = x) < 2 \int_{-\infty}^{\sqrt{\frac{1-\rho}{1+\rho}}x} \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}z^2}}{\Phi(\lambda x)} dz.$$

Since both $\Phi(2\theta x)$ and $\Phi(\lambda x) \rightarrow 1$ as $x \rightarrow -\infty$ we have

$$2P(X_2 \leq x|X_1 = x) \sim 2 \int_{-\infty}^{\sqrt{\frac{1-\rho}{1+\rho}}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 2\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}x\right)$$

$$\text{so} \quad 2P(X_2 \leq F_2^{-1}(u)|X_1 = F_1^{-1}(u)) \sim 2\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}F_1^{-1}(u)\right),$$

since $F_1^{-1}(u) \rightarrow -\infty$ as $u \rightarrow 0^+$. Since $F_1^{-1}(u) \sim y(u)$ as $u \rightarrow 0^+$ from Theorem 1 and using (11), we can prove $2\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}F_1^{-1}(u)\right) \sim 2\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}y(u)\right)$ as $u \rightarrow 0^+$ by showing $[F_1^{-1}(u)]^2 - [y(u)]^2 \rightarrow 0$, as $u \rightarrow 0^+$. This is in turn equivalent to showing

$$e^{[F_1^{-1}(F_1(z))]^2 - [y(F_1(z))]^2} = e^{z^2 + 2\log(\frac{F_1(z)}{2})\sqrt{-4\pi\log(\frac{F_1(z)}{2})\sqrt{2\pi}}} \rightarrow 1, \quad \text{as } z \rightarrow -\infty.$$

Then

$$\begin{aligned}
& e^{z^2 + 2 \log(\frac{F_1(z)}{2}) \sqrt{-4\pi \log(\frac{F_1(z)}{2} \sqrt{2\pi})}} \\
& \sim e^{z^2} \times \left(\frac{1}{\sqrt{2\pi}} |z|^{-1} e^{-z^2/2}\right)^2 \times (-4\pi \log(|z|^{-1} e^{-z^2/2})), \quad \text{by (10)} \\
& = 1 + \frac{4\pi \log |z|}{2\pi z^2} \rightarrow 1, \quad \text{as } z \rightarrow -\infty.
\end{aligned}$$

This implies that

$$2P(X_2 \leq F_2^{-1}(u) | X_1 = F_1^{-1}(u)) \sim 2\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}} y(u)\right) \sim \sqrt{\frac{1+\rho}{1-\rho}} (-\pi \log u)^{-\frac{\rho}{1+\rho}} u^{\frac{1-\rho}{1+\rho}}$$

as $u \rightarrow 0^+$, by using (11) and (18). Finally, by using (32) we have

$$P(X_2 \leq F_2^{-1}(u) | X_1 \leq F_1^{-1}(u)) \sim \frac{1+\rho}{2} \sqrt{\frac{1+\rho}{1-\rho}} (-\pi \log u)^{-\frac{\rho}{1+\rho}} u^{\frac{1-\rho}{1+\rho}}, \quad \text{as } u \rightarrow 0^+.$$

□

The theorem shows that when $\theta < 0$ there is minimal difference between symmetric and skew normal in terms of the intermediate tail dependence as they share the same regular varying index which is $\frac{1-\rho}{1+\rho}$. On the other hand, when $\theta > 0$, it has a larger regular varying index by a factor of $(1 + 2\theta^2(1 + \rho))$ when compared to the normal case and therefore skew normal has smaller intermediate tail dependence than the normal in the lower tail.

We shall finish the paper by briefly discussing the corresponding result for the upper tail dependence.

Corollary 1. *Let $\mathbf{X} \sim SN_2(\boldsymbol{\theta}, R)$ with $\theta_1 = \theta_2 = \theta$. In self evident notation and as $u \rightarrow 1^-$,*

(a) *if $\theta < 0$ (so $-\theta = |\theta| > 0$),*

$$\begin{aligned}
\lambda_U(u) &= P(X_1 \geq F_1^{-1}(u) | X_2 \geq F_2^{-1}(u)) \\
&\sim (1-u)^{\beta^2} \frac{\alpha^3}{\pi \lambda^4 \beta (1 + \beta^2)^2} \sqrt{\frac{2}{\pi}} (2\pi \lambda)^{1+\beta^2} \left(\frac{1+\lambda^2}{2}\right)^{\frac{3}{2}} [-\log(1-u)]^{\beta^2 - \frac{1}{2}}
\end{aligned}$$

$$\text{with } \lambda = \frac{|\theta|(1+\rho)}{\sqrt{1+\theta^2(1-\rho^2)}}, \alpha = \frac{|\theta|(1+\rho)}{\sqrt{1+2\theta^2(1+\rho)}} \text{ and } \beta = \sqrt{\frac{(1-\rho)(1+2\theta^2(1+\rho))}{1+\rho}};$$

(b) *if $\theta > 0$ (so $-\theta < 0$),*

$$\lambda_U(u) \sim (1-u)^{\frac{1-\rho}{1+\rho}} \times \frac{1+\rho}{2} \sqrt{\frac{1+\rho}{1-\rho}} (-\pi \log(1-u))^{-\frac{\rho}{1+\rho}}.$$

Proof. If $\mathbf{Y} = (Y_1, Y_2)^T \stackrel{d}{=} -\mathbf{X}$ where $\mathbf{X} = (X_1, X_2)^T$ with continuous marginal distributions, then from Lemma 1 of Fung and Seneta (2014) (with self-evident notation), we have $\lambda_U^{\mathbf{X}}(u) = \lambda_L^{\mathbf{Y}}(1-u)$. The proof is completed by noting that when $\mathbf{X} \sim SN_2(\boldsymbol{\theta}, R)$, we have $\mathbf{Y} = -\mathbf{X} \sim SN_2(-\boldsymbol{\theta}, R)$ from Azzalini and Dalla Valle (1996) and applying Theorem 2. □

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